4.1 Maximum and Minimum Values

Learning Objectives: After completing this section, we should be able to

- find absolute extrema and local extrema of a function via its derivative.

Here are some questions we are trying to answer:

- How many items should a manufacturer make to

  maximize profit.

- What trajectory of an object

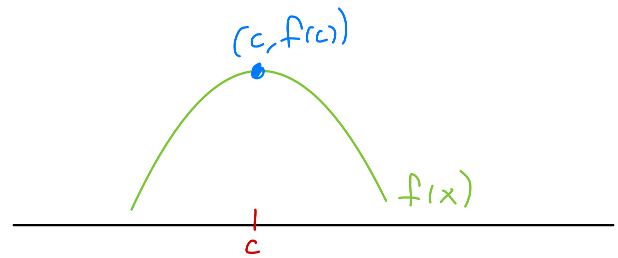
  would be optimal when travelling to a certain height

- What position gives

  max effeciancy

Here are some informal definitions. See the textbook for the formal definition if you are interested.

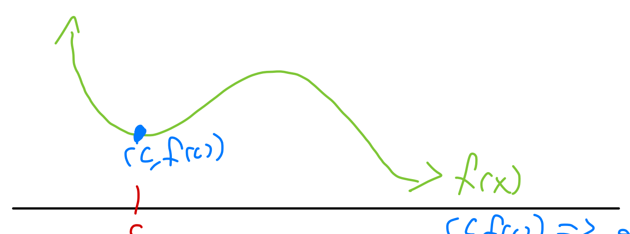
The point $(c, f(c))$ is a:



- (relative) **local maximum** if

  $f(c) \geq f(x)$

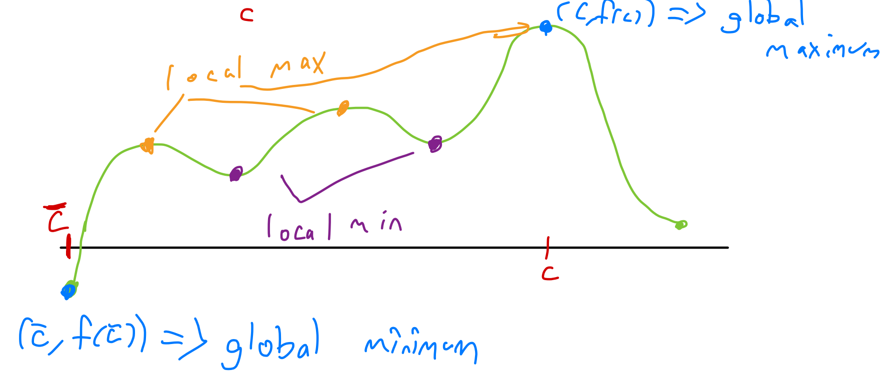
  for all x near c on both sides of c



- (relative) **local minimum** if

  $f(c) \leq f(x)$

  for all x near c on both sides of c



- **absolute max** or **global max** if

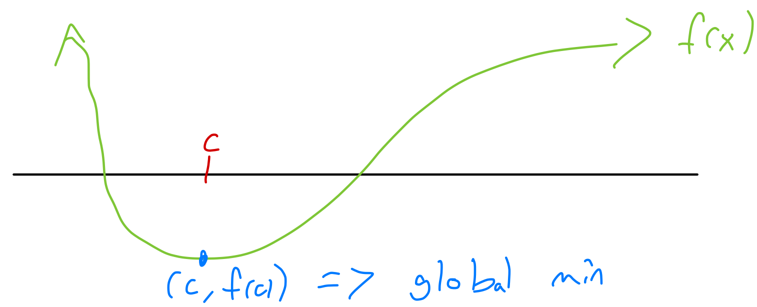
  $f(c) \geq f(x)$

  for all x in the domain of f

- **absolute min** or **global min** if

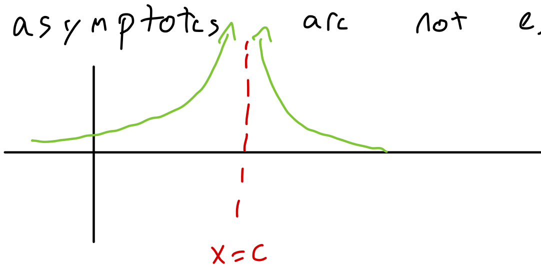
  $f(c) \leq f(x)$

  for all x in the domain of f



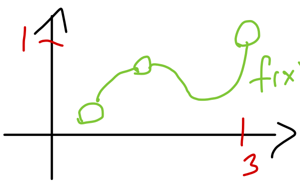
There are some fringe cases

- asymptotes are not extreme values, as $f(x)$ is not defined



$\Rightarrow$ no max at $x=c$

- holes or $f(x)$ is defined on an open interval

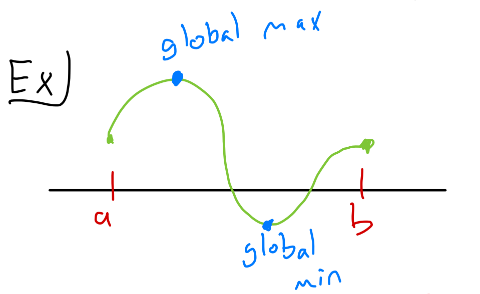


$\lim_{x\to 3^-} f(x) = 1$, but $f(1)$ is undefined, so it can't be a max value

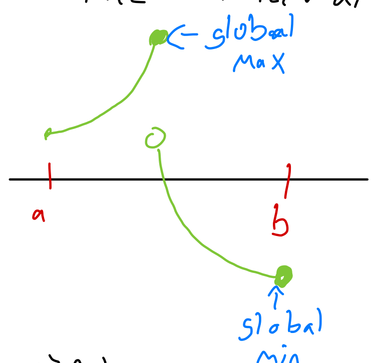
Theorem (Extreme Value Theorem). *If $f(x)$ is continuous on a closed interval $[a,b]$, then*

f has an absolute minimum value and an absolute maximum value on the interval $[a,b]$.

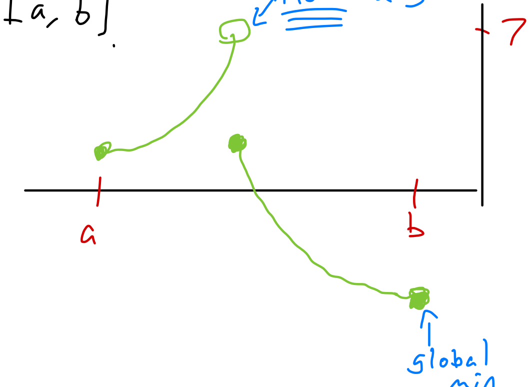
Ex)



$\Rightarrow$ only example of these 3 where the Extreme Value thm applies



$\Rightarrow$ not continuous on $[a,b]$, but global extrema exist

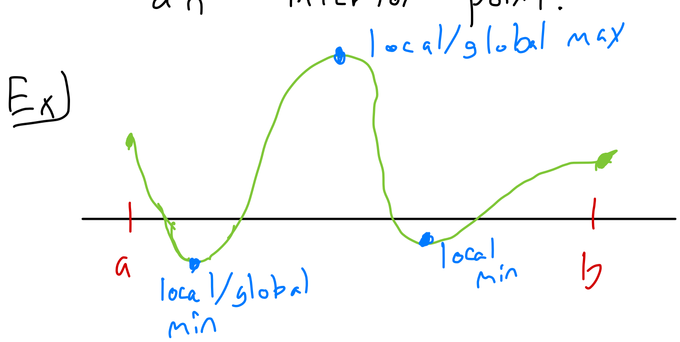


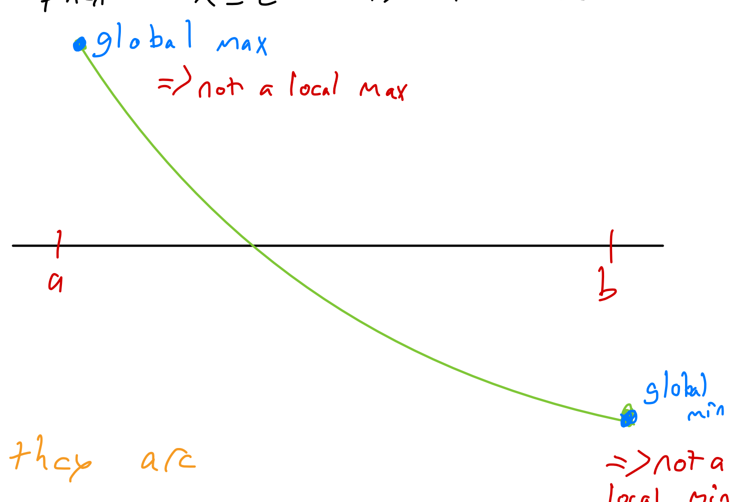
$\Rightarrow$ not continuous on $[a,b]$, but only the global min exists.

Note: Local minimums and maximums must be

at interior points; i.e., $a < c < b$, then $x=c$ is the location of an interior point.

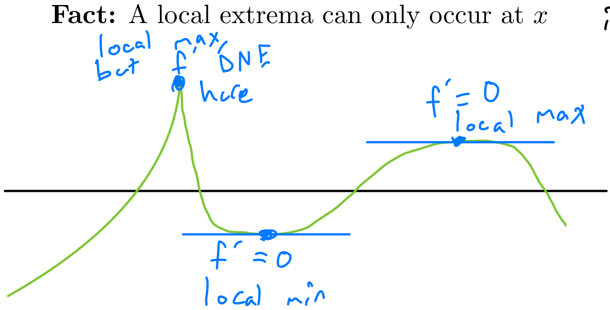
Ex)





Endpoints are not local extrema, as they are not interior points. However, they are potentially global extrema.

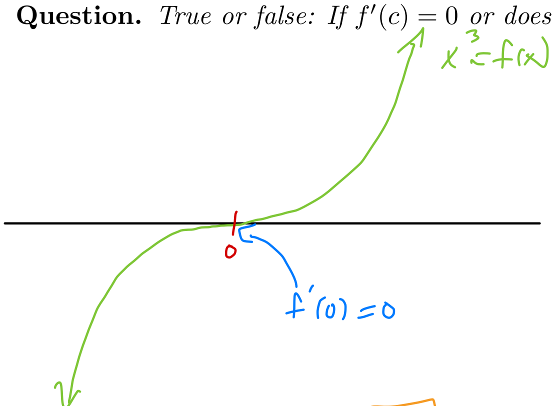
Fact: A local extrema can only occur at x if $f'(x)=0$ or if $f'(x)$ DNE

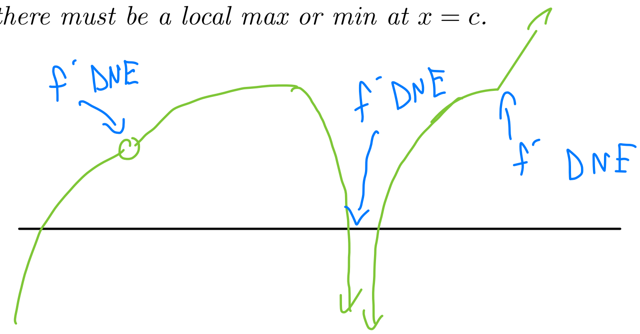


Definition. If c is an interior point in our domain; i.e., $a < c < b$, then c is called a critical value/number of f if $f'(c)=0$ or $f'(c)$ DNE

Question. *True or false: If $f'(c) = 0$ or does not exist, then there must be a local max or min at $x = c$.*

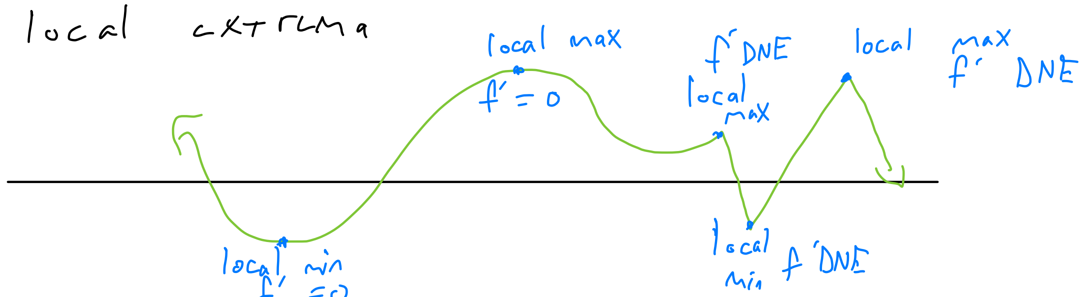
False





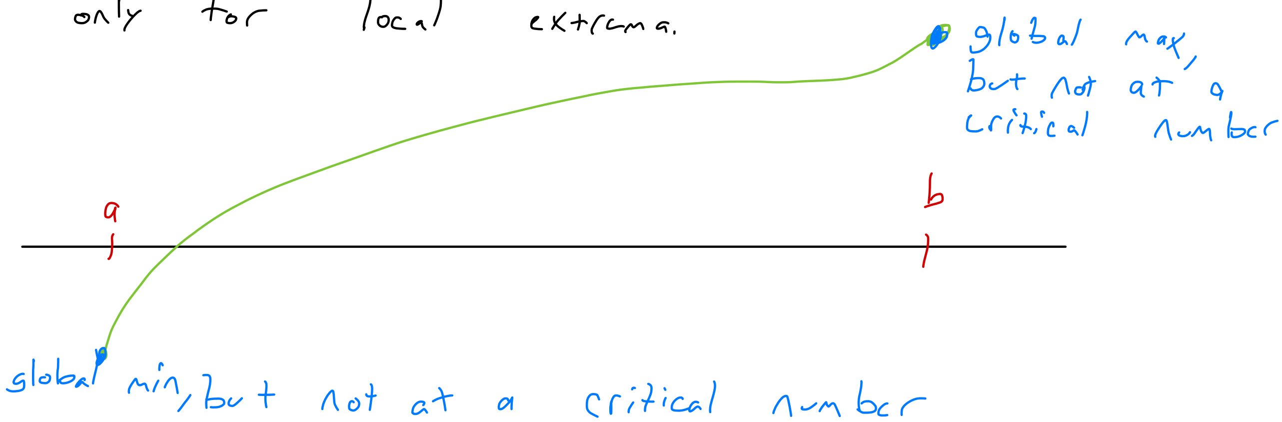
Question. *True or false: If a $\boxed{\text{local}}$ max or min occurs at $x = c$, then $f'(c) = 0$ or does not exist.*

True! This is the "FACT", noting local maxes & mins are local extrema



Question. *True or false: Maximums or minimums only occur at critical values.*

False: The question ignores absolute/global extrema. This is true only for local extrema.



Fact: If $f(x)$ is continuous on $[a, b]$, then, by the Extreme Value Theorem, global extrema exist. These two extrema either occur at a critical number or an endpoint.

Example. Let $f(x) = e^x \sin(x)$ on $[-2, 7]$. Find the absolute maximum and minimum.

- Check critical numbers and endpoints

$f(x) = e^x \cdot \sin(x)$

$f'(x) = e^x \cdot \cos(x) + e^x \cdot \sin(x)$ $\Rightarrow$ there is no bad behavior, so the derivative exists for all x in $[-2, 7]$

To find critical numbers, set $f'(x) = 0$

$$e^x \cdot \cos(x) + e^x \cdot \sin(x) = 0$$

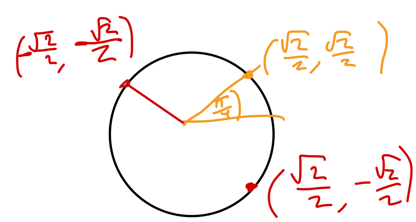
$$\Rightarrow \underbrace{e^x}_{\text{never } 0} \left[\cos(x) + \sin(x)\right] = 0 \Rightarrow \cos(x) + \sin(x) = 0$$

$$\Rightarrow \frac{\sin(x)}{\cos(x)} = \frac{-\cos(x)}{\cos(x)}$$

$$\Rightarrow \tan(x) = \frac{\sin(x)}{\cos(x)} = -1$$

So, using a calculator, $\tan(x) = -1$ for x in $[-2, 7]$ if

$x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$ $\leftarrow$ critical numbers



Check endpoints and critical numbers

$f(x) = e^x \sin(x)$

$f(-2) = e^{-2}\sin(-2) \approx -0.125$

$f(-\frac{\pi}{4}) = e^{-\frac{\pi}{4}} \sin(-\frac{\pi}{4}) \approx -0.3224$

$f(\frac{3\pi}{4}) = e^{\frac{3\pi}{4}} \sin(\frac{3\pi}{4}) \approx 7.4605$

$f(\frac{7\pi}{4}) = e^{\frac{7\pi}{4}} \sin(\frac{7\pi}{4}) \approx -172.64$ $\leftarrow$ global min value is approximately -172.64 at $x = \frac{7\pi}{4}$

$f(7) = e^7 \cdot \sin(7) \approx 720.47$ $\leftarrow$ global max value is approximately 720.47 located at $x = 7$

We need to be careful with our terminology.

maximum

- The global maximum value is approximately 720.47 (y-coordinate)
- The global maximum occurs at or is located at $x = 7$ (x-coordinate)
- The global max is the point, i.e., an x- and y-value; $\approx \underbrace{(7, 720.47)}_{\text{point}}$

minimum

- The global minimum value is approximately -172.64 (y-coordinate) located at $x = \frac{7\pi}{4}$ (x-coordinate)

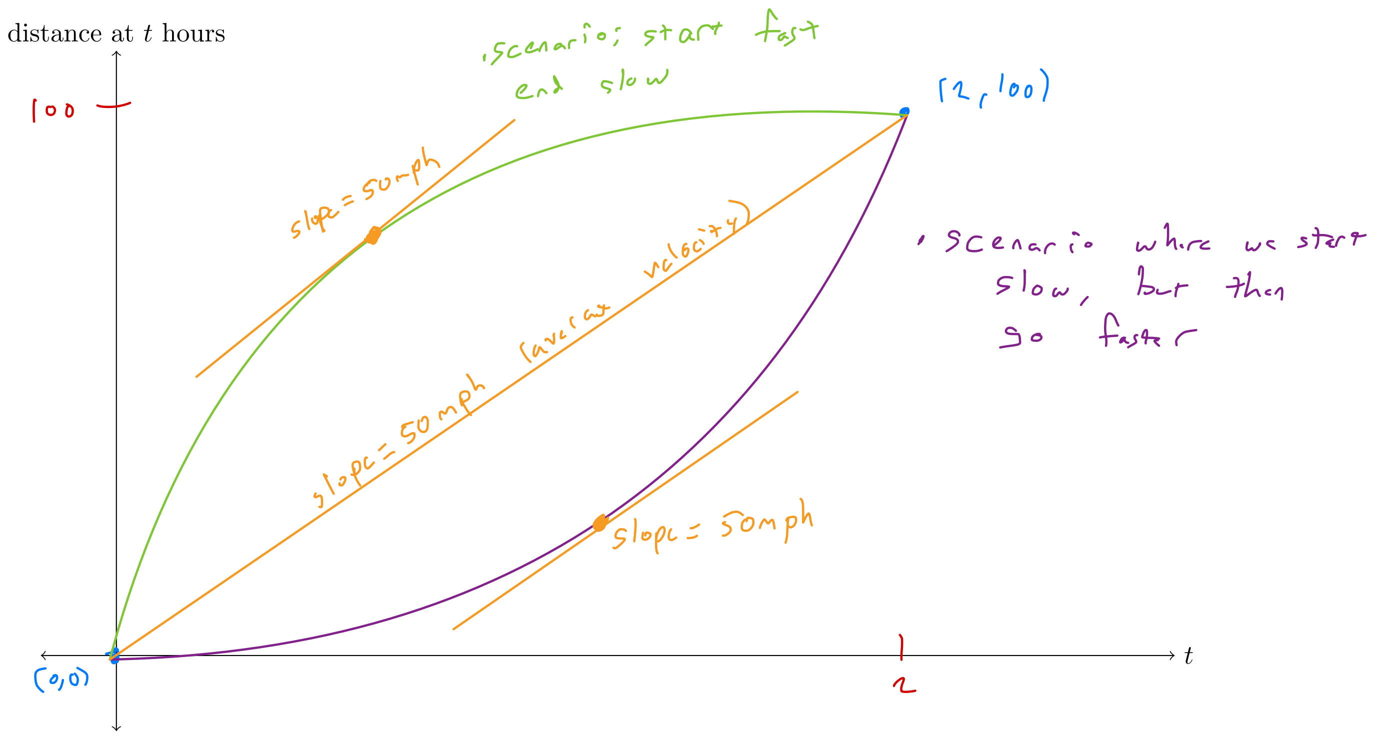
  The global minimum is approximately $\underbrace{\left(\frac{7\pi}{4}, -172.64\right)}_{\text{point}}$

4.2 The Mean Value Theorem

Learning Objectives: After completing this section, we should be able to

- use the Mean Value Theorem and apply it to prove other results.

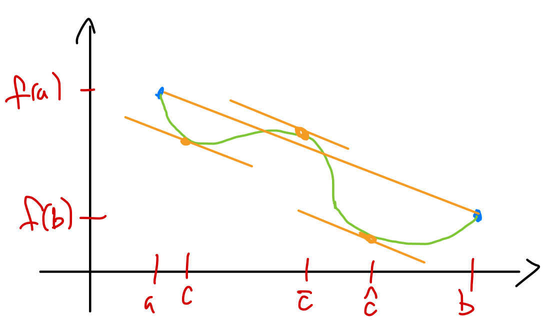
Example. Suppose you are driving on a highway. You note that you have travelled 100 miles in the last 2 hours. What do you know about your instantaneous velocity at any point on the trip?



At some point, the instantaneous velocity must be 50 mph.
If you start out slow, then go fast, then at some point you were travelling exactly at 50 mph.

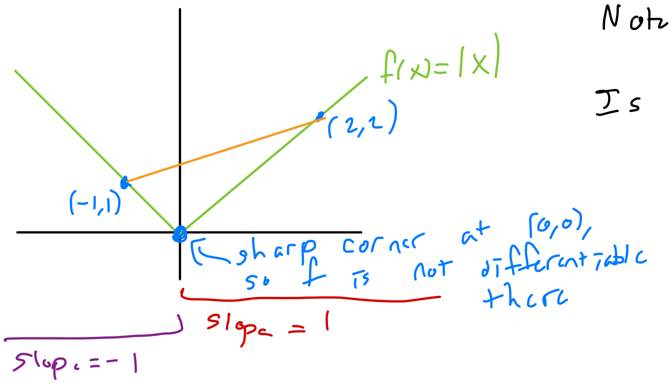
Theorem (Mean Value Theorem). *If f is continuous over $[a, b]$ and* differentiable on (a, b), then, there is at least one number c in (a, b) such that the instantaneous rate of change of f at c is the average rate change of f over $[a, b]$; ie.,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



All of c, $\bar{c}$, and $\hat{c}$ are possible numbers from the Mean Value Theorem.

Example. Let $f(x) = |x|$. Note $f(-1) = 1$ and $f(2) = 2$.



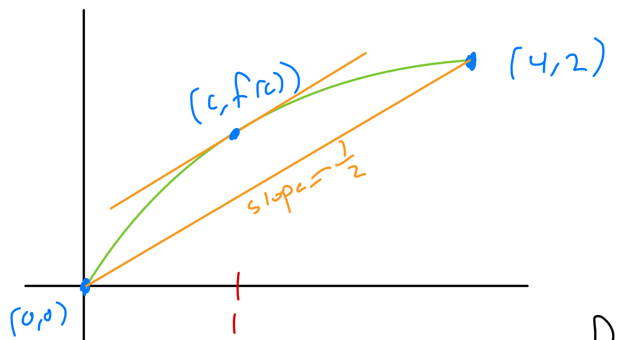
Note, the $AROC = \dfrac{f(2)-f(-1)}{2-(-1)} = \dfrac{2-1}{3} = \dfrac{1}{3}$

Is there a c in $(-1,2)$ such that $f'(c) = \frac{1}{3}$?

No. The Mean Value Theorem requires f to be continuous on $[-1,2]$ and differentiable on $(-1,2)$

↑ violated at $x=0$, so MVT does not apply

Example. Let $f(x) = \sqrt{x}$. Note $f(0) = 0$ and $f(4) = 2$.



$AROC = \dfrac{f(4)-f(0)}{4-0} = \dfrac{2-0}{4} = \dfrac{1}{2}$

Is there a c in $(0,4)$ such that $f'(c) = \frac{1}{2}$?

Does the MVT apply?
- f is continuous on $[0,4]$
- f is differentiable on $(0,4)$

Yes!

Let's find c. $f'(c) = \dfrac{f(4)-f(0)}{4-0} = \dfrac{1}{2}$. Note, $f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow \frac{1}{2}(c)^{-\frac{1}{2}} \cdot c^{\frac{1}{2}} = \frac{1}{2} \cdot c^{\frac{1}{2}}$

$\Rightarrow 2 \cdot \frac{1}{2} = \frac{1}{2}c^{\frac{1}{2}} \cdot 2$

$\Rightarrow 1 = c^{\frac{1}{2}} \Rightarrow c = 1^2 = 1$. So, $c=1$ is where $f'(c) = \frac{1}{2}$

Example. Suppose f is continuous on $[a,b]$ and differentiable on (a,b) where $f(a) = f(b)$.

Then, the MVT guarantees there is a c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{0}{b-a} = 0$

This is Rolle's Theorem

$\Rightarrow$ Typically, we prove Rolle's theorem first, and then use this to prove MVT.

Example. In Iowa, there are marks on the interstate highway every 0.1 miles visible to an airplane or helicopter. The speed limit on the interstate is 70 mph. A police helicopter notices that a car crosses one mark and then 4.8 seconds later the car crosses the next mark. Will the driver get a speeding ticket?

Car travelled 0.1 miles in 4.8 seconds.

Let $d(t)$ be the distance at time t between the car and the initial mark.

$\Rightarrow d(0) = 0$ miles

$$4.8 \text{ seconds} \cdot \frac{1 \text{ min}}{60 \text{ seconds}} \cdot \frac{1 \text{ hour}}{60 \text{ min}} = \frac{1}{750} \text{ hours}$$

$\Rightarrow d\left(\frac{1}{750}\right) = 0.1$ miles.

Note, $d(t)$ is continuous on $\left[0, \frac{1}{750}\right]$, and differentiable on $\left(0, \frac{1}{750}\right)$. The MVT ~~OK~~ promises there there is a c in $\left(0, \frac{1}{750}\right)$ such that

$$d'(c) = \frac{d\left(\frac{1}{750}\right) - d(0)}{\frac{1}{750} - 0} = \frac{0.1 - 0}{\frac{1}{750}} = 75 \text{ mph}$$

The car gets a speeding ticket!

4.3 Derivatives and Shapes of Graphs

Learning Objectives: After completing this section, we should be able to

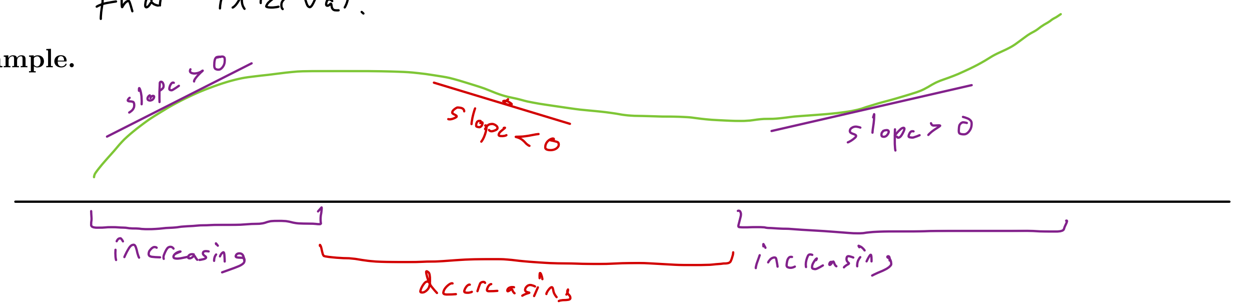
- find the intervals of increase and decrease of a function using the first derivative of the function.
- find the intervals of concavity of a function using the second derivative of the function.

4.3.1 First Derivative

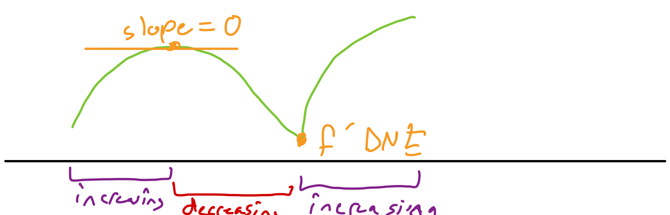
Definition. *Increasing/Decreasing*

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

Example.



Fact: The only places f changes from increasing to decreasing or decreasing to increasing is when $f'(c)=0$ or when f' DNE



The first derivative test for local extrema: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) except at critical numbers, then

- If $f'(x)$ changes from positive to negative at $x=c$, then there is a local maximum located at $x=c$
- If $f'(x)$ changes from negative to positive at $x=c$, then there is a local minimum located at $x=c$
- If $f'(x)$ doesn't change sign at $x=c$, then there is no extrema at $x=c$.

Example. Let $f(x) = 2x^3 + 3x^2 - 12x + 1$. Find all local extrema.

1) Identify critical numbers; i.e., where is $f'(x) = 0$ or $f'(x)$ DNE?

$$f'(x) = 6x^2 + 6x - 12, \quad \text{note } f'(x) \text{ exists everywhere}$$
$$= 6[x^2 + x - 2]$$
$$= 6[(x+2)(x-1)] \overset{(set)}{=} 0$$

So, $f'(x) = 0$ when $x = -2$ and when $x = 1$. These are the critical numbers

2) Determine intervals of increasing and decreasing.

- Draw a number line and label it
- Test values between critical numbers to determine the sign of $f'(x)$

only x-values where f' can change sign

sign chart for $f'(x) = 6(x+2)(x-1)$

always label!

$+$	$x=-2$	$-$	$x=1$	$+$
try $x=-3$		try $x=0$		try $x=2$
$f'(-3) = 6(-3+2)(-3-1)$		$f'(0) = 6(0+2)(0-1)$		$f'(2) = 6(2+2)(2-1)$
$(+)(-)(-)$		$(+)(+)(-)$		$(+)(+)(+)$
> 0		< 0		> 0

3) Interpret the sign chart and apply first derivative test

Note, $f'(x) > 0$ on $(-\infty, -2)$ and $(1, \infty)$,

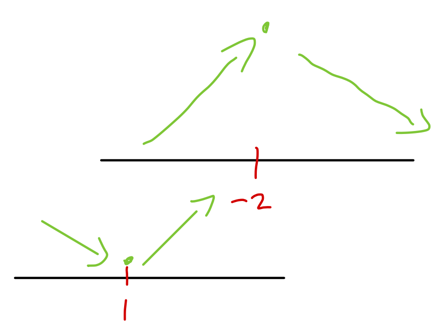
So $f(x)$ is increasing on $(-\infty, -2) \cup (1, \infty)$

Note, $f'(x) < 0$ on $(-2, 1)$, so $f(x)$ is decreasing on $(-2, 1)$

1st deriv test:

$\Rightarrow$ local max at $x = -2$

$\Rightarrow$ local min at $x = 1$



local max value at $x = -2$ is $f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 21$

local min value at $x = 1$ is $f(1) = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$

You try!

Example. Let $f(x) = 3x^5 - 20x^3$. Find all local extrema.

1) $\Rightarrow f'(x) = 15x^4 - 60x^2$ (note, $f'(x)$ exists everywhere)

$= 15x^2[x^2 - 4]$

$= 15x^2[(x+2)(x-2)]$

$f'(x) = 0$ if $x = 0, -2, 2$, so $x = -2, 0, 2$ are critical numbers

2) sign chart for $f'(x) = 15x^2(x+2)(x-2)$

$+$	-2	$-$	0	$-$	2	$+$
try $x=-3$		try $x=-1$		try $x=1$		try $x=3$
$f'(-3) = 15(-3)^2(-3+2)(-3-2)$		$f'(-1) = 15(-1)^2(-1+2)(-1-2)$		$f'(1) = 15(1)^2(1+2)(1-2)$		$f'(3) = 15(3)^2(3+2)(3-2)$
$(+)(+)(-)(-)$		$(+)(+)(+)(-)$		$(+)(+)(+)(-)$		$(+)(+)(+)(+)$
>0		<0		<0		>0

3) $f(x)$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 0) \cup (0, 2)$

$f(x)$ has a local max at $x = -2$ with value $f(-2) = 3(-2)^5 - 20(-2)^3 = 64$,

and a local min at $x = 2$ with value $f(2) = 3(2)^5 - 20(2)^3 = -64$

(no extrema at $x=0$)

Example. Let $f(x) = x^2 - 2\ln(x)$. Find all local extrema.

1) $f'(x) = 2x - 2\frac{1}{x}$. Find when

$f'(x) = 0$ and $f'(x) = 2x - \frac{2}{x}$ DNE

$2x - \frac{2}{x} = 0$ $\Rightarrow x = 0$, $f'(x)$ DNE

$+\frac{2}{x}$ $+\frac{2}{x}$

$\Rightarrow x \cdot 2x = \frac{2}{x} \cdot x$

$\Rightarrow 2x^2 = 2$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ cause $f'(x) = 0$

$\Rightarrow$ It appears our critical numbers are $x = -1, 0, 1$

Note, $f(x)$ is only defined for $x > 0$, as $-2\ln(x)$ term is undefined otherwise

$\Rightarrow$ the only critical number we need to consider is $x = 1$

2)

is a important number, but not a critical number to consider

sign chart for $f'(x) = 2x - \frac{2}{x}$

$x=0$ $-$ $x=1$ $+$

try $x = \frac{1}{2}$ try $x = 2$

$f'(\frac{1}{2}) = 2(\frac{1}{2}) - \frac{2}{(\frac{1}{2})}$ $f'(2) = 2(2) - \frac{2}{2}$

$= 1 - 4 < 0$ $= 4 - 1 = 3 > 0$

3) So $f(x) = x^2 - 2\ln(x)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

Also, f has a local min at $x = 1$

and it's min value is

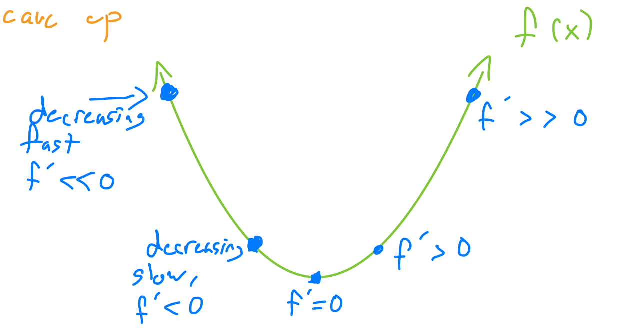
$f(1) = 1^2 - 2\ln(1) = 1$

4.3.2 Second Derivative

Consider the second derivative: $f''(x)$, which is the first derivative of $f'(x)$

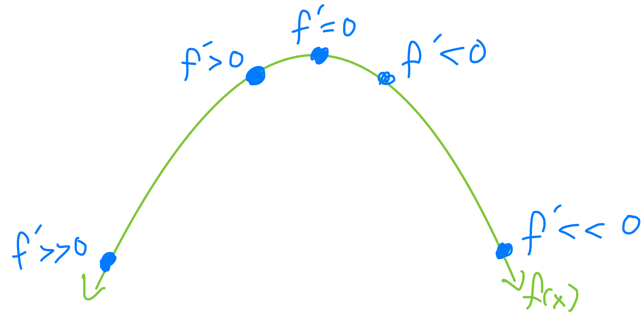
Definition. If $f'(x)$ is increasing, then f is concave up

$\Rightarrow$ Equivalently: If $f''(x) > 0$, then f is concave up



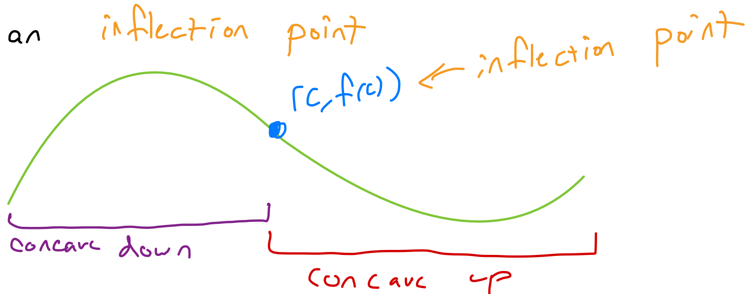
Definition. If $f'(x)$ is decreasing, then f is concave down

$\Rightarrow$ If $f''(x) < 0$, then f is concave down



Definition. If f changes concavity at $x = c$, then

$(c, f(c))$ is an inflection point



Example. Determine the intervals of concavity for $f(x) = x^4 - 2x^3 + 1$.

What is the sign of $f''(x)$?

1)
$$f'(x) = 4x^3 - 6x^2$$
$$f''(x) = 12x^2 - 12x$$
$$= 12x(x-1) \overset{\text{(set)}}{=} 0$$

(note, f'' exists everywhere)

so $x=0$ and $x=1$ are potential inflection points

2) Make sign chart for $f''(x)$

sign chart for $f''(x) = 12(x)(x-1)$

+	$x=0$	−	$x=1$	+
try $x=-1$		try $x=\frac{1}{2}$		try $x=2$
$f''(-1) = 12(-1)(-1-1)$		$f''(\frac{1}{2}) = 12(\frac{1}{2})(\frac{1}{2}-1)$		$f''(2) = 12(2)(2-1)$
(+)(−)(−)		(+)(+)(−)		(+)(+)(+)
>0		<0		>0

Example Continued.

3) Analyze sign chart

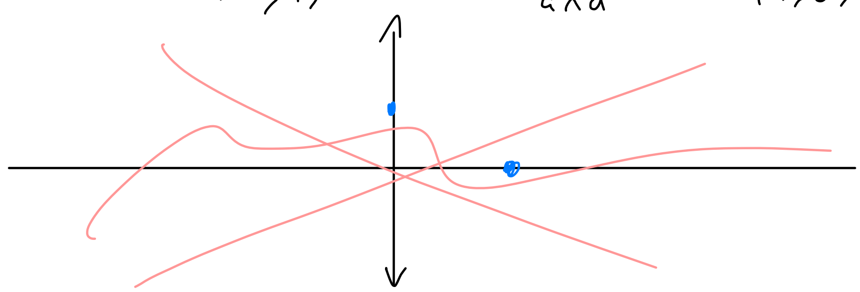
$f(x)$ is concave up on $(-\infty, \cancel{-1}\ 0) \cup (1, \infty)$
and is concave down on $(\cancel{-1}\ 0, 1)$.

There are inflection points located at $x = \cancel{-1}\ 0$ and $x = 1$

Inflection points: $(\cancel{-1}\ 0, \cancel{f(-1)}\ f(0))$ and $(1, f(1))$

$\Rightarrow (0, 0^4 - 2\cdot 0^3 + 1)$ and $(1, 1^4 - 2\cdot 1^3 + 1)$

$\Rightarrow (0, 1)$ and $(1, 0)$



You try!

Example. Find the intervals of concavity and the inflection points for $f(x) = 2x^4 + 8x^3 + 12x^2 - x - 2$.

1) $f'(x) = 8x^3 + 24x^2 + 24x - 1$

$f''(x) = 24x^2 + 48x + 24$, note f'' exists everywhere

$= 24[x^2 + 2x + 1]$

$= 24[(x+1)(x+1)] \overset{(set)}{=} 0$

So $x = -1$ is a possible inflection point

2) sign chart for $f''(x) = 24(x+1)^2$

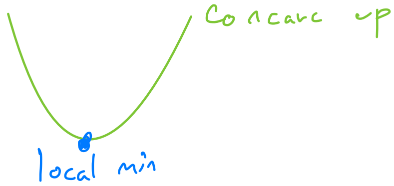
$+$ | $x=-1$ | $+$

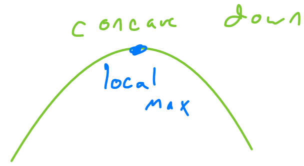
try $x = -2$: $f''(-2) = 24(-2+1)^2 = (+)(+) > 0$

try $x = 0$: $f''(0) = 24(0+1)^2 = (+)(+) > 0$

3) $f(x)$ has no inflection points and is concave up on $(-\infty, -1) \cup (-1, \infty)$

Second derivative test for local extrema: What can you say about extrema when $f(x)$ is concave up? Down?





Second derivative test: Suppose $(c, f(c))$ is a critical point. Then,

- If $f''(c) > 0$, then

  $(c, f(c))$ is a local minimum

- If $f''(c) < 0$, then

  $(c, f(c))$ is a local maximum

- If $f''(c) = 0$, then no information $\Rightarrow$ use first derivative test

Example. From before: Let $f(x) = 2x^3 + 3x^2 - 12x + 1$. Use the second derivative test to find extrema.

$$f'(x) = 6x^2 + 6x - 12$$, note it exists everywhere

$$= 6[x^2 + x - 2]$$

$$= 6[(x+2)(x-1)] \overset{(set)}{=} 0$$

So $x = -2$ and $x = 1$ are critical numbers

Recall $f'(x) = 6x^2 + 6x - 12$

$$f''(x) = 12x + 6$$

$x = -2$: $f''(-2) = 12(-2) + 6 = -18 < 0 \Rightarrow$ local max of f at $x = -2$

$x = 1$: $f''(1) = 12(1) + 6 = 18 > 0 \Rightarrow$ local min of f at $x = 1$

You try!

Example. Let $f(x) = 2x^2 \ln(x) - 11x^2$. Find all local extrema using the second derivative test.

$$f'(x) = \left[(4x)\cdot \ln(x) + (2x^2)\cdot\left(\tfrac{1}{x}\right)\right] - 22x$$

$$= 4x \cdot \ln(x) + 2x - 22x$$

$$= 4x \cdot \ln(x) - 20x$$

$$= 4x\left[\ln(x) - 5\right] \overset{(\text{set})}{=} 0$$

$x = 0$ is a critical number and so is the solution to

$$\ln(x) - 5 = 0 \Rightarrow \ln(x) = 5$$
$$\Rightarrow x = e^5$$

can ignore, as $f(0)$ is undefined

Recall $f'(x) = 4x\cdot\ln(x) - 20x$

$$\Rightarrow f''(x) = \left[(4)\cdot \ln(x) + 4x\left(\tfrac{1}{x}\right)\right] - 20$$

$$f''(e^5) = \left[4 \cdot \ln(e^5) + 4\right] - 20$$

$$= [20 + 4] - 20 = 4$$

$$> 0$$

so f has a local min at $x = e^5$

with value $f(e^5) = 2(e^5)^2 \cdot \ln(e^5) - 11(e^5)^2$

Question. *Should you use the first derivative test or the second derivative test?*

- If $f''(x)$ is easy to find; i.e., no complicated rules necessary, then the second derivative test is also quicker
- If $f''(x)$ requires a complex rule to find, then use the first derivative test

(First derivative always works)

4.4 Indeterminate Forms and L'Hopital's Rule

Learning Objectives: After completing this section, we should be able to

- apply L'Hopital's Rule to evaluate the limit of an expression in an indeterminate form.

Recall that

$\frac{\infty}{\infty}$ and $\frac{0}{0}$ are indeterminate forms

numerator -> ∞
denominator -> ∞

numerator -> 0
denominator -> 0

Remember, when talking about indeterminate forms,

it is always in the context of limits

(sometimes spelled L'Hospital)

Theorem (L'Hopital's Rule). *If* $\lim_{x\to c} \frac{f(x)}{g(x)}$ *is* yielding $\frac{0}{0}$ or $\frac{\infty}{\infty}$ indeterminate forms then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

if the second limit exists

Important!

- This is not

  the Quotient rule

- Only applies

  if the theorem's conditions hold; i.e.,
  the original limit has an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example. $\lim_{x\to\infty} \frac{x+1}{x^2-5}$ (this is 0, as the degree is higher in denominator than numerator)

explaining the work

$$\lim_{x\to\infty} \frac{x+1}{x^2-5} \underset{(\frac{\infty}{\infty})}{\overset{(LH)}{=}} \lim_{x\to\infty} \frac{1+0}{2x-0} = \lim_{x\to\infty} \frac{1}{2x} = 0$$

justify which indeterminate form applies

- everytime you apply L'Hopital's rule, state you are using it (LH) and justify it (write the applicable indeterminate form)

Example. $\lim_{x\to 0} \frac{\sin(x)}{x}$

Note $\sin(0) = 0$

$x\big|_{x=0} = 0$

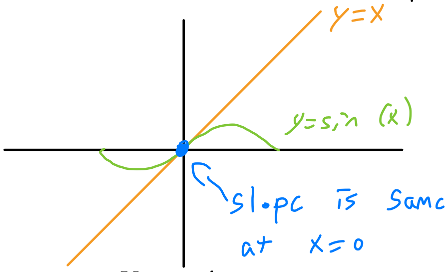
$$\lim_{x\to 0} \frac{\sin(x)}{x} \overset{(LH)}{\underset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to 0} \frac{\cos(x)}{1}$$

$\frac{d}{dx}\sin(x) = \cos(x)$

$\frac{d}{dx}x = 1$

$$= \frac{\cos(0)}{1} = 1$$

What does this imply? Both $\sin(x)$ and x go to 0 at same rate when evaluating the limit



- For x near 0, $\sin(x)$ and x are roughly the same

You try!

Example. $\lim_{x\to 0} \frac{\sin(7x)}{4x}$

$$\lim_{x\to 0} \frac{\sin(7x)}{4x} \overset{(LH)}{\underset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to 0} \frac{\cos(7x)\cdot 7}{4}$$

$$= \frac{\cos(7\cdot 0)\cdot 7}{4}$$

$$= \frac{1\cdot 7}{4} = \frac{7}{4}$$

You try!

Example. $\lim_{x\to\infty} \frac{3x^2+2x-1}{8x^2+100}$

$$\lim_{x\to\infty} \frac{3x^2+2x-1}{8x^2+100} \overset{(LH)}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{6x+2}{16x}$$

$$\overset{(LH)}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{6}{16} = \frac{6}{16} = \frac{3}{8}$$

Example. $\lim_{x\to 0} \frac{x-2}{x^2+4} = \frac{0-2}{0^2+4} = -\frac{2}{4} = -\frac{1}{2}$

Not indeterminate, so we cannot use L'Hopital's rule.

try to use LH even though it doesn't apply

~~$\overset{(LH)}{=} \lim_{x\to 0} \frac{1}{2x}$ DNE as in the form $\frac{\text{nonzero constant}}{0}$~~

Example. $\lim_{x\to 0} \frac{x\sin(x)}{1-\cos(x)} \to \frac{0\cdot\sin(0)}{1-\cos(0)} \to \frac{0}{1-1} \to \frac{0}{0}$

$$\lim_{x\to 0} \frac{x\cdot\sin(x)}{1-\cos(x)} \underset{\left(\frac{0}{0}\right)}{\overset{(LH)}{=}} \lim_{x\to 0} \frac{x\cdot\cos(x)+1\cdot\sin(x)}{0 - -\sin(x)}$$

$$= \lim_{x\to 0} \frac{x\cdot\cos(x)+\sin(x)}{\sin(x)} \to \frac{0\cdot\cos(0)+\sin(0)}{\sin(0)} \to \frac{0\cdot 1+0}{0} \to \frac{0}{0}$$

$$\underset{\left(\frac{0}{0}\right)}{\overset{(LH)}{=}} \lim_{x\to 0} \frac{[x\cdot(-\sin(x))+1\cdot\cos(x)]+\cos(x)}{\cos(x)}$$

$$= \frac{0\cdot(-\sin(0))+\cos(0)+\cos(0)}{\cos(0)} = \frac{0+1+1}{1}$$

$$= 2$$

use product rule/chain rule if necessary when using LH

Example. $\lim_{x\to\infty} \frac{\ln(x)}{2\sqrt{x}} = \lim_{x\to\infty} \frac{\ln(x)}{2x^{\frac{1}{2}}} \to \frac{\infty}{\infty}$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{(LH)}{=}} \lim_{x\to\infty} \frac{1/x}{2\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}} = \lim_{x\to\infty} \frac{x^{-1}}{x^{-\frac{1}{2}}} \left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right)$$

Not pleasant, simplify first

$$= \lim_{x\to\infty} \frac{x^{-1+\frac{1}{2}}}{1} = \lim_{x\to\infty} x^{-\frac{1}{2}}$$

$$= \lim_{x\to\infty} \frac{1}{x^{\frac{1}{2}}} = 0$$

You try!

Example. $\lim_{x\to\infty} \frac{x}{(\ln(x))^2}$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{(LH)}{=}} \lim_{x\to\infty} \frac{1}{2\cdot(\ln(x))'\cdot\frac{1}{x}} \left(\frac{x}{x}\right)$$

$\frac{d}{dx}(\ln(x))^2 = 2\cdot(\ln(x))'\cdot\frac{1}{x}$ by the chain rule.

$$= \lim_{x\to\infty} \frac{x}{2\cdot\ln(x)\cdot 1}$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{(LH)}{=}} \lim_{x\to\infty} \frac{1}{2\cdot\left(\frac{1}{x}\right)} \frac{(x)}{(x)}$$

$$= \lim_{x\to\infty} \frac{x}{2} \to \infty$$

Diverges to ∞.

DNE

What about other indeterminate forms? $0\cdot\infty$, 0^0, ∞^∞ are indeterminate when considering limits

Consider the indeterminate form $0\cdot\infty$:

- Need to modify it into a fraction yielding an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (can't change problem)

- Note:

$0 \to \frac{1}{\infty}$ in limit form

$\infty \to \frac{1}{0}$ in limit form

move a term ~~in the numerator~~ into the denominator of the denominator

Example. $\lim_{x\to 0^+} x\ln(x)$ $\to$ $0(-\infty)$ form as $x\to 0$ and $\ln(x)\to-\infty$ when $x\to 0^+$

Two options

$$= \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln(x)}}$$

$\frac{0}{0}$

or

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}$$

$\frac{\infty}{\infty}$

$\Rightarrow$ Easier to apply LH

$$\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \underset{\left(\frac{\infty}{\infty}\right)}{\overset{LH}{=}} \lim_{x\to 0^+} \frac{1/x}{(-1)x^{-2}} \left(\frac{x^2}{x^2}\right) = \lim_{x\to 0^+} \frac{x^2/x}{(-1)} = \lim_{x\to 0^+} -x$$

$$= -0 = 0$$

You try!

Example. $\lim_{x\to\infty} e^{-x}x^2$

Note $e^{-x} \to 0$ as $x \to \infty$

$x^2 \to \infty$ as $x \to \infty$

$$= \lim_{x\to\infty} \frac{x^2}{\frac{1}{e^{-x}}} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{\overset{(LH)}{=}} \lim_{x\to\infty} \frac{2x}{e^x} \underset{\left(\frac{\infty}{\infty}\right)}{\overset{(LH)}{=}} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

$\lim_{x\to\infty} \frac{e^{-x}}{x^{-2}}$ will keep getting worse when applying LH

Example. $\lim_{x\to\frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right)\tan(x)$

$\frac{\pi}{2} - x \to 0$

$\tan(x) \to \infty$

as $x \to \frac{\pi}{2}$

(If $\tan(x)$, $\cot(x)$, $\sec(x)$, $\csc(x)$ appear in a limit, then convert to sine and cosine to start)

$$= \lim_{x\to\frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right)\frac{\sin(x)}{\cos(x)} = \underbrace{\lim_{x\to\frac{\pi}{2}^-} \frac{\left(\frac{\pi}{2} - x\right)\cdot \sin(x)}{\cos(x)}} \to \frac{0\cdot\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)} \to \frac{0\cdot 1}{0}$$

Apply LH directly to

Not on Exam 2

4.7 Optimization Problems

Learning Objectives: After completing this section, we should be able to

- convert an optimization problem in words into a mathematical optimization problem.
- solve an optimization problem.

Now that we have tools to find extrema, we can use them to solve real-world problems!

Example. Suppose x and y are 2 numbers. Find these two positive numbers satisfying the equation $\boxed{xy = 3}$ and the $\boxed{\text{sum } x + 2y \text{ is as small as possible.}}$

$x>0,\ y>0$ (sign restriction)

$xy=3$ (constraint equation)

minimum of $x+2y$ (objective function)

$\Rightarrow$ min $z = x+2y$

satisfying the constraint and sign restrictions

- Write the objective fnctn in terms of only 1 variable

$\Rightarrow$ use the constraint equation to do this

$xy = 3$

$\Rightarrow y = \frac{3}{x}$

$\Rightarrow$ Replace all y's in objective fnctn

$x + 2y = x + 2\left(\frac{3}{x}\right) = x + \frac{6}{x} = f(x)$

Minimize $f(x) = x + \frac{6}{x}$, for $x>0$

critical numbers are when $f'(x) = 0$ or $f'(x)$ DNE

$f(x) = x + \frac{6}{x} = x + 6x^{-1}$

$\Rightarrow f'(x) = 1 + 6(-1)x^{-2}$

$= 1 - \frac{6}{x^2}$

Note, $f'(x)$ DNE when $x=0$. Set $f'(x) = 1 - \frac{6}{x^2} = 0$

$\Rightarrow 1 = \frac{6}{x^2} \Rightarrow 1\cdot x^2 = 6$

$\Rightarrow x = \pm\sqrt{6}$

So $x = -\sqrt{6},\ 0,\ \boxed{\sqrt{6}}$ are critical numbers

only one satisfying sign restriction

Use 2nd derivative test to determine if $x=\sqrt{6}$ is location of min.

$$f'(x) = 1 - 6x^{-2}$$

$$f''(x) = -6(-2)x^{-3} = \frac{12}{x^3}$$

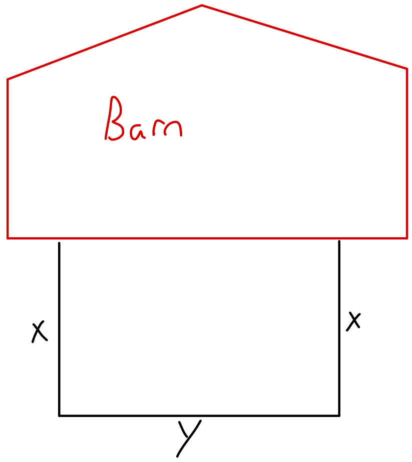
Evaluate $f''(\sqrt{6}) = \frac{12}{(\sqrt{6})^3} > 0 \Rightarrow x = \sqrt{6}$ is location of min.

Still need to give y: $y = \frac{3}{x}$ $\Leftarrow$ came from constraint eqn

$$\Rightarrow y = \frac{3}{\sqrt{6}}$$

The optimal numbers are $x = \sqrt{6}$ and $y = \frac{3}{\sqrt{6}}$

Example. A rectangular pen is being built against the side of a barn. There is 1000 m of fencing available. What dimensions of the pen maximize the area of the pen?



Implicit sign restrictions $x>0,\ y>0$

Constraint: 1000 m fence available

total fence used $= x+y+x$

$$2x+y=1000$$

objective fnctn:

maximize area = (length)(width)

$= (x)(y)$

$$A = x\cdot y$$

$2x+y=1000$

$\Rightarrow\ y = 1000-2x$

$A = xy = x(1000-2x)$

So maximize $A = x(1000-2x)$

$\Rightarrow\ A = 1000x - 2x^2$

$A' = 1000-4x$, note A' exists everywhere

$1000 - 4x \overset{(\text{set})}{=} 0$

$\Rightarrow\ 1000 = 4x$

$\Rightarrow\ x = \frac{1000}{4} = 250$

So $x = 250$ m is a critical number

Sign chart for $A' = 1000-4x$

+	250	−
try $x=1$ $\Rightarrow A' = 1000-4\cdot 1 > 0$		try $x=1000$ $A' = 1000 - 4\cdot 1000 < 0$

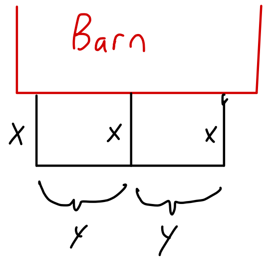
So $x=250$ m is location of max

Recall $y = 1000-2x$

$\Rightarrow\ y = 1000-2(250) = 500$ m

The optimal dimensions of the pen is 250 m of fence perpendicular to barn and 500 m of fence parallel to barn.

Example. A rancher is building 2 adjacent, rectangular pens against a barn, each with an area of 50 m^2. What are the dimensions of each pen that minimize the amount of fence that must be used?



Sign restrictions: $x > 0,\ y > 0$

Constraint: Area $= 50\ m^2$

$A = x \cdot y, \implies 50 = x \cdot y$

Objective fnctn:

minimize fence used

total fence $= x + y + x + y + x = 3x + 2y$

$50 = x \cdot y$

$\implies y = \frac{50}{x}$

$F = 3x + 2y = 3x + 2\left(\frac{50}{x}\right)$

$= 3x + \frac{100}{x}$

$= 3x + 100x^{-1}$

$F' = 3 + 100(-1)x^{-2}$

$= 3 - \frac{100}{x^2}$, Note F' DNE if $x = 0$

$F' = 3 - \frac{100}{x^2} \overset{(set)}{=} 0$

$\implies 3 = \frac{100}{x^2}$

$\implies 3x^2 = 100$

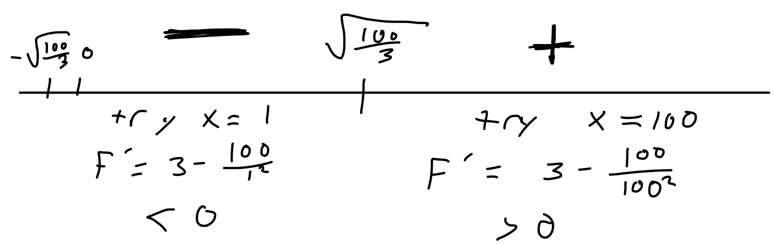
$\implies x^2 = \frac{100}{3}$

$\implies x = \pm\sqrt{\frac{100}{3}}$

critical numbers are $x = -\sqrt{\frac{100}{3}},\ 0,\ \underbrace{\sqrt{\frac{100}{3}}}_{\text{only one satisfying } x>0}$

sign chart for

$F' = 3 - \frac{100}{x^2}$



so $x = \sqrt{\frac{100}{3}}$ is location of min

$y = \frac{50}{x} \implies y = \frac{50}{\sqrt{\frac{100}{3}}}$

The optimal dimensions are $\sqrt{\frac{100}{3}}$ m of fence ~~parallel~~ perpendicular to barn and $\frac{50}{\sqrt{\frac{100}{3}}}$ m of fence ~~perpendicular~~ parallel to barn, for each pen.

4.9 Antiderivatives

Learning Objectives: After completing this section, we should be able to

- find antiderivatives of given functions.

We've spent the majority of the semester taking derivatives. How do we undo taking a derivative?

Definition. An **antiderivative** of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$; i.e.,

$$F'(x) = f(x)$$

They are not unique!

Example. $\frac{d}{dx}(2x^3+4) = 6x^2$.

So, an antiderivative for $6x^2$ is $2x^3+4$

but so is $2x^3-9$

and $2x^3+\pi \cdot e$

$\vdots$

etc...

Antiderivatives come in in a 1-parameter family.

$F(x) = 2x^3 + C$ is the antiderivative for $f(x) = 6x^2$, where C is a parameter that is any constant number

Don't separate the antiderivative from its family

Notation:

integrand (find the antiderivative of this)

$$\underbrace{F(x)}_{\text{antiderivative function}} = \underbrace{\int}_{\text{integral symbol}} \boxed{f(x)}\, \underbrace{dx}_{\text{differential}}$$

(indicates the variable of integration)

"F(x) is the antiderivative of f(x) with respect to x."

or

"F(x) is the indefinite integral of f(x) with respect to x."

Example. $\int \boxed{6x^2}\boxed{dx} = 2x^3 + C$

Integrand: $6x^2$

Differential: dx

Antiderivative: $2x^3+C$

Finding an antiderivative is called integrating, just as finding a derivative is called differentiating.

Whenever you have the integral symbol $\int$, you always need a corresponding differential.

Let's find antiderivatives for basic functions.

1. **Powers**

Recall $\frac{d}{dx} x^p = p \cdot x^{p-1}$

1) multiply by the power
2) subtract 1 from the power

To undo:

1) Add 1 to power
2) Divide by the new power

$$\int x^p dx = \frac{1}{p+1} x^{p+1} + C, \quad \text{for } p \neq -1$$

Example. $\int x^5 dx$ power is $p=5$

$$= \frac{1}{5+1} x^{5+1} + C = \frac{1}{6} x^6 + C$$

Double check

$$\frac{d}{dx}\left(\frac{1}{6} x^6 + C\right) = \frac{6}{6} x^5 + 0 = x^5 \checkmark$$

2. **Constant Multiple Rule**

Recall $\frac{d}{dx} a \cdot f(x) = a \frac{d}{dx} f(x) = a \cdot f'(x)$,

where a is any constant

$\Rightarrow$ same holds for integration

$$\int a \cdot f(x) dx = a \int f(x) dx, \quad \text{where } a \text{ is any constant}$$

Example. $\int 6x^2 dx = 6 \int x^2 dx$

$$= 6\left(\frac{1}{2+1} x^{2+1} + C\right)$$

$$= 6\left(\frac{1}{3} x^3 + C\right)$$

$$= 6 \cdot \frac{1}{3} x^3 + 6 \cdot C$$

$$= 2x^3 + C$$

technically, this is a new C! It still represents adding an arbitrary constant, and 6 times an arbitrary constant is still arbitrary

3. **Sum Rule**

Recall $\frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x)+g'(x)$

$$\int (f(x)+g(x))dx = \int f(x)\,dx + \int g(x)\,dx$$

Example. $\int (6x^2+x^5)dx = \int 6x^2 dx + \int x^5 dx$

$= (\overset{2}{\cancel{6}}x^3 + C) + (\frac{1}{6}x^6 + \bar{C})$

$= 2x^3 + \frac{1}{6}x^6 + C$

absorb all arbitrary constants into 1

4. **Trig Rules**

- $\int \cos(x)dx = \sin(x) + C$
- $\int \sin(x)dx = -\cos(x) + C$

  Verify $\frac{d}{dx}(-\cos(x) + C) = -(-\sin(x)) + 0 = \sin(x)$ ✓
- $\int \sec^2(x)dx = \tan(x) + C$
- $\int \csc^2(x)dx = -\cot(x) + C$
- $\int \sec(x)\tan(x)dx = \sec(x) + C$
- $\int \csc(x)\cot(x)dx = -\csc(x) + C$

5. **Inverse Trig**

- $\int \frac{1}{1+x^2}dx = \arctan(x) + C$
- $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$
- $\int -\frac{1}{\sqrt{1-x^2}}dx = \arccos(x) + C$

6. **Logs and Exponentials**

Recall: • $\frac{d}{dx} e^x = e^x$ $\Rightarrow$ • $\int e^x dx = e^x + C$

• $\frac{d}{dx} b^x = b^x \cdot \ln(b)$ (for $b > 0$) $\Rightarrow$ • $\int b^x dx = \frac{1}{\ln(b)} \cdot b^x + C$

• $\frac{d}{dx} \ln(|x|) = \frac{1}{x}$ $\Rightarrow$ • $\int \frac{1}{x} dx = \ln(|x|) + C$

Note $\int \frac{1}{x} dx = \int x^{-1} dx \neq \frac{1}{-1+1} x^{-1+1} + C$

nonsense! Can't divide by 0

7. **Constant Chains**

Ex] $\int \sin(3x) dx = \frac{1}{\boxed{3}}(-\cos(3x)) + C$

• accounts for "3x"

• only works for constant multiples

$\Rightarrow$ Doubt check: $\frac{d}{dx}\left(\frac{1}{3}(-\cos(3x)) + C\right)$

$f(u) = -\frac{1}{3}\cos(u)$ $g(x) = 3x$

$f'(u) = -\frac{1}{3}\sin(u)\cdot(-1)$ $g'(x) = 3$

$= f'(g(x)) \cdot g'(x) = f'(3x) \cdot 3 = \cancel{-}\cancel{\frac{1}{3}} \sin(3x) \cdot (\cancel{-}1) \cdot \cancel{3} = \sin(3x)$

Rule: If $F(x) = \int f(x) dx$, then $\int f(a \cdot x) dx = \frac{1}{a} \cdot F(a \cdot x) + C$, for constant a,

Caution: Nonexample: $\int \sin(2x^2) dx \neq \cancel{\frac{1}{4x} \cdot \cos(2x^2) + C}$

Note $2x^2$ is not a constant multiple of x

Example. $\int \left(x + 14 - \sqrt{x^3} + 3x^{-6} - \frac{2}{x} + \cos(4x) - \sec^2(6x)/8 + \frac{1}{3}e^{-x} - \frac{4}{1+x^2} + \pi^x\right) dx$

$+ \cos(4x) +$

$= \int [x + 14 \cdot 1 + (-1) \cdot x^{3/2} + 3 \cdot x^{-6} + (-1) \cdot 2 \cdot x^{-1} \cancel{+} -\frac{1}{8} \cdot \sec^2(6x) + \frac{1}{3} e^{(-1)\cdot x} + (-4) \frac{1}{1+x^2} + \pi^x] dx$

$\int x\, dx = \frac{1}{1+1} x^{1+1} + c = \frac{1}{2}x^2 + c$

$\int 14 \cdot 1 dx = \int 14 \cdot x^0 dx = 14 \frac{1}{0+1} x^{0+1} + C = 14x + C$

$\int (-1) x^{3/2} dx = (-1) \int x^{3/2} dx = (-1) \frac{1}{\frac{3}{2}+1} x^{\frac{3}{2}+1} + c = -\frac{2}{5} x^{5/2} + c$

$\int 3 \cdot x^{-6} dx = 3 \frac{1}{-6+1} x^{-6+1} + c = -\frac{3}{5} x^{-5} + c$

$\int (-1)(2) x^{-1} dx = -2 \int \frac{1}{x} dx = -2 \cdot \ln(|x|) + C$

$\int \cos(4x) dx = \frac{1}{4} \sin(4x) + c$

$\int -\frac{1}{8} \sec^2(6x) dx = -\frac{1}{8} \int \sec^2(6x) dx = -\frac{1}{8} \cdot \frac{1}{6} \tan(6x) + C = -\frac{1}{48} \tan(6x) + C$

$\int \frac{1}{3} e^{(-1)x} dx = \frac{1}{3} \frac{1}{-1} e^{(-1)\cdot x} + c = -\frac{1}{3} e^{-x} + c$

$\int (-4) \frac{1}{1+x^2} dx = -4 \cdot \arctan(x) + c$

$\int \pi^x dx = \frac{\pi^x}{\ln(\pi)} + C$

$= \frac{1}{2}x^2 + 14x - \frac{2}{5}x^{5/2} - \frac{3}{5}x^{-5} - 2 \cdot \ln(|x|) + \frac{1}{4}\sin(4x) - \frac{1}{48}\tan(6x) - \frac{1}{3}e^{-x} - 4\arctan(x) + \frac{\pi^x}{\ln(\pi)} + C$